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Completing Bethe's Equations at Roots of Unity

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In a previous paper we demonstrated that Bethe's equations are not sufficient to specify the eigenvectors of the XXZ model at roots of unity for states where the Hamiltonian has degenerate eigenvalues. We here find the equations which will complete the specification of the eigenvectors in these degenerate cases and present evidence that the sl_2 loop algebra symmetry is sufficiently powerful to determine that the highest weight of each irreducible representation is given by Bethe's ansatz.

KEY WORDS: Bethe's ansatz; loop algebra; quantum spin chain; XXZ-model.

1. INTRODUCTION

The XXZ model with periodic boundary conditions, defined by

$$H = -\frac{1}{2} \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \cos \gamma \sigma_j^z \sigma_{j+1}^z), \tag{1.1}$$

where σ_j^i is the i Pauli spin matrix at site j and $j = L+1 \equiv 1$, has long been studied by means of what has come to be called Bethe's equation

$$\left(\frac{\sinh \frac{1}{2}(v_j + i\gamma)}{\sinh \frac{1}{2}(v_j - i\gamma)} \right)^L = \prod_{\substack{l=1 \\ l \neq j}}^{\frac{L}{2} - |S^z|} \frac{\sinh \frac{1}{2}(v_j - v_l + 2i\gamma)}{\sinh \frac{1}{2}(v_j - v_l - 2i\gamma)} \tag{1.2}$$

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where $S^z=\frac{1}{2}\sum_{j=1}^{L}\sigma_j^z$ is a conserved quantum number. The eigenvalues of (1.1) are given by

$$E=\frac{\cos\gamma}{2}L-2\sum_{j=1}^{\frac{L}{2}-|S^z|}\frac{\sin^2\gamma}{\cosh v_j-\cos\gamma} \tag{1.3}$$

and the eigenvectors are given in terms of the v_j.

The authors who originally derived these equations[1–6] concentrated on the energy and eigenvector of the ground state and this computation has been done with great rigor and clarity. However, the derivations seem to superficially extend to all eigenstates and for almost three decades there have been extensive efforts[4, 7–28] made to study these excited energy states and for generic values of γ it has been proven[21–23] that the solutions to (1.2) do in fact give all the 2^L eigenvalues of (1.1).

On the other hand it was shown by Bethe[1] in his original study of the case $\gamma=0$ that for the Heisenberg antiferromagnet the equation (1.2) with finite v_j does not give all eigenvalues. Furthermore it was realized[29] as early as 1973 that the equation (1.2) will not specify solutions of the form where both the numerator and denominator simultaneously vanish and that these special solutions do indeed occur in the "root of unity case"

$$\gamma_0=\frac{r}{N}\pi \tag{1.4}$$

where r and N are relatively prime and $1\leqslant r\leqslant N-1$. Therefore there are indeed values of γ which are not generic in the sense of refs. 21–23 where Bethe's equation (1.2) with finite v_j is known not to give all eigenstates of the system.

Recently it was demonstrated[30] that when the root of unity condition (1.4) holds that the Hamiltonian (1.1) has an invariance under the sl_2 loop algebra and that this symmetry leads to degenerate multiplets of energy eigenvalues. Moreover we subsequently showed[31] when $\gamma\to\gamma_0$ that there are solutions v_k to (1.2) which have N members of the form

$$v_k=\alpha+i2k\gamma_0 \qquad \text{with} \quad 1\leqslant k\leqslant N \tag{1.5}$$

where α is in general complex with $-\pi/N\leqslant \mathrm{Im}\alpha\leqslant\pi/N$. We referred to these solutions as exact complete N strings and in ref. 31 we numerically presented many examples. These solutions give a contribution of zero to the energy independent of α by virtue of the identity

$$\sum_{k=1}^{N}\frac{1}{\cosh(\alpha+ik2\pi r/N)-\cos r\pi/N}=0 \tag{1.6}$$

and thus are responsible for degeneracies in the energy eigenvalue spectrum.

These exact complete N strings all have the property that in the equation (1.2) they give rise to factors $0/0$. Therefore the equation (1.2) will not be able to determine the parameter α of the exact complete N string (1.5). It is for this reason that in ref. 31 we said that Bethe's equation is incomplete at roots of unity.

However, it is clear that the missing equations can be obtained in principle by setting

$$\gamma = \gamma_0 + \epsilon \tag{1.7}$$

in (1.2) and carefully taking the limit $\epsilon \to 0$. It is the purpose of this paper to carry out this construction which will complete the specification of the Bethe's roots v_k.

For a fixed value of $S^z \geqslant 0$ any solution v_k will have $1 \leqslant k \leqslant \frac{L}{2} - S^z$. If the solution contains n of the complete exact N strings there will be n_o other roots where

$$n_o = \frac{L}{2} - S^z - nN \tag{1.8}$$

We denote these roots as v_k^0. We will call these roots "ordinary" roots because we will show that they satisfy the Bethe's equation (where we will let L be even)

$$\left(\frac{\sinh \frac{1}{2}(v_j^0 + i\gamma_0)}{\sinh \frac{1}{2}(v_j^0 - i\gamma_0)} \right)^L = \prod_{\substack{l=1 \\ l \neq j}}^{n_o} \frac{\sinh \frac{1}{2}(v_j^0 - v_l^0 + 2i\gamma_0)}{\sinh \frac{1}{2}(v_j^0 - v_l^0 - 2i\gamma_0)}. \tag{1.9}$$

This equation for v_k^0 does not involve the parameters α_j of the exact complete N strings and by definition does not have any ambiguous terms of the form $0/0$. These ordinary roots v_j^0 may be real or may have imaginary parts which are organized into strings (see ref. 31 for many examples). For the purposes of this paper this information is not needed and is thus not indicated in the notation.

The simplest case with complete exact N strings is the state which does not contain any ordinary roots. For a chain of length L these special states have $S^z = \frac{L}{2} - nN$ where n is the number of exact complete N strings. The n parameters α_m are determined from the n equations

$$\sum_{k=0}^{N-1} \sinh^L \tfrac{1}{2}(\alpha_m + (2k+1)\, i\gamma_0)$$
$$\times \left(L \coth \tfrac{1}{2}\,(\alpha_m + (2k+1)\, i\gamma_0) - 2 \sum_{\substack{l=1\\ l\neq m}}^{n} \sum_{j=0}^{N-1} \coth \tfrac{1}{2}(\alpha_m - \alpha_l + i2j\gamma_0) \right) = 0 \tag{1.10}$$

In general when there are ordinary roots in the state the n parameters α_m are determined from

$$\sum_{k=0}^{N-1} \sinh^L \tfrac{1}{2}(\alpha_m + (2k+1)\, i\gamma_0) \prod_{l=2k+4}^{N+2k+1} P_l(\alpha_m)$$
$$\times \left(L \coth \tfrac{1}{2}\,(\alpha_m + (2k+1)\, i\gamma_0) - 2 \sum_{\substack{l=1\\ l\neq m}}^{n} \sum_{j=0}^{N-1} \coth \tfrac{1}{2}\,(\alpha_m - \alpha_l + i2j\gamma_0) \right.$$
$$\left. - \sum_{l=1}^{n_o} (\coth \tfrac{1}{2}(\alpha_m - v_l^0 + 2ki\gamma_0) + \coth \tfrac{1}{2}(\alpha_m - v_l^0 + 2(k+1)\, i\gamma_0)) \right) = 0 \tag{1.11}$$

where

$$P_k(\alpha_m) = \prod_{l=1}^{n_o} \sinh \tfrac{1}{2}(\alpha_m - v_l^0 + 2ik\gamma_0) \tag{1.12}$$

and we note the periodicity $P_{k+N}(\alpha_k) = P_k(\alpha_k)$.

In the special case $N = 2$ the product $\prod_l P_l(\alpha_m)$ is replaced by one and setting

$$z_m = \exp(\alpha_m), \qquad \zeta_m = \exp(v_m^0) \tag{1.13}$$

the equation (1.11) reduces to

$$L(z_m^2+1)\frac{(z_m+i)^{L-2} + (i)^L\,(z_m-i)^{L-2}}{(z_m+i)^L + (i)^L\,(z_m-i)^L} - 4 \sum_{\substack{l=1\\ \neq m}}^{n} \frac{z_m^2+z_l^2}{z_m^2-z_l^2} - 2 \sum_{l=1}^{n_o} \frac{z_m^2+\zeta_l^2}{z_m^2-\zeta_l^2} = 0. \tag{1.14}$$

It is to be noted that this equation is not the equation which is obtained by replacing the righthand side of (1.2) by $(-1)^{L/2-|S^z|-1}$ which is what would be obtained if we "formally" set $\gamma = \pi/2$. We also note that the case $N = 2$ has been treated in ref. 32 in quite a different context.

These results will be derived in section 2. We will see that in order to determine the parameters α_j we need to expand (1.2) to order ϵ^2 and that the equation (1.11) is obtained as a consistency condition arising from the vanishing of the secular determinant of a set of homogeneous linear equations.

We will discuss the consequences of our result and the possible relations with the evaluation parameters introduced in ref. 31 in Section 3. In particular we present compelling evidence that for $N \geqslant 3$ the sl_2 loop algebra is powerful enough to determine not only the structure of the degenerate multiplets but the highest weight vectors themselves. On the other hand the highest weight vectors are already known from the Bethe's ansatz solution of the XXZ spin chain. Therefore we conclude that the Bethe's equation (1.2) is in fact contained in the irreducible representations of $U(\hat{sl}_2)$ at level zero.

2. DERIVATION OF (1.11)

We begin our derivation by first considering (1.2) directly at $\gamma = \gamma_0$ where v_j is an ordinary root v_j^0 and split the product over l on the right hand side into the n_o contributions from ordinary roots and the nN contributions from the N strings to find

$$\left(\frac{\sinh \frac{1}{2}(v_j^0 + i\gamma_0)}{\sinh \frac{1}{2}(v_j^0 - i\gamma_0)}\right)^L$$

$$= \prod_{\substack{l=1 \\ l \neq j}}^{n_o} \frac{\sinh \frac{1}{2}(v_j^0 - v_l^0 + 2i\gamma_0)}{\sinh \frac{1}{2}(v_j^0 - v_l^0 - 2i\gamma_0)} \prod_{m=1}^{n} \prod_{k=1}^{N} \frac{\sinh \frac{1}{2}(v_j^0 - \alpha_m - 2i(k-1)\gamma_0)}{\sinh \frac{1}{2}(v_j^0 - \alpha_m - 2i(k+1)\gamma_0)}. \tag{2.1}$$

The product over k gives unity independent of α_m and thus (2.1) reduces to (1.9) as desired.

In order to derive (1.11) we cannot directly set $\gamma = \gamma_0$ in (1.2) and let $v_j = v_{j,k} = \alpha_j + 2ik\gamma_0$ because factors of $0/0$ will occur on the right hand side. Therefore we first write (1.2)

$$\sinh^L \tfrac{1}{2}(v_{j,k} - i\gamma)\, q(v_{j,k} + 2i\gamma) + \sinh^L \tfrac{1}{2}(v_{j,k} + i\gamma)\, q(v_{j,k} - 2i\gamma) = 0 \tag{2.2}$$

with

$$q(v_{j,k} \pm 2i\gamma) = \prod_{i=1}^{n_o} \sinh \tfrac{1}{2}(v_{j,k} - v_i^0 \pm 2i\gamma) \prod_{l=1}^{n} \prod_{m=1}^{N} \sinh \tfrac{1}{2}\,(v_{j,k} - v_{l,m} \pm 2i\gamma). \tag{2.3}$$

We then define for the roots which become exact N strings

$$v_{j,k} = \alpha_j + 2ik\gamma_0 + \epsilon v^{(1)}_{j,k} + \epsilon^2 v^{(2)}_{j,k}. \tag{2.4}$$

and for all other ordinary roots

$$v_l = v^0_l + \epsilon v^{(1)}_l. \tag{2.5}$$

We proceed in several steps.

Expansion of $\sinh^L \frac{1}{2}(v_{j,k} \pm i\gamma)$

We expand $\sinh^L \frac{1}{2}(v_{j,k} \pm i\gamma)$ to order ϵ as

$$\begin{aligned}
&\sinh^L \tfrac{1}{2}(v_{j,k} \pm i\gamma)\\
&\quad\sim \sinh^L \tfrac{1}{2}(\alpha_j + 2ik\gamma_0 + \epsilon v^{(1)}_{j,k} \pm i(\gamma_0 + \epsilon))\\
&\quad\sim \sinh^L \tfrac{1}{2}[\alpha_j + (2k \pm 1)\, i\gamma_0](1 + \epsilon L \tfrac{1}{2}\,(v^{(1)}_{j,k} \pm i) \coth \tfrac{1}{2}\,[\alpha_j + (2k \pm 1)\, i\gamma_0]).
\end{aligned} \tag{2.6}$$

Expansion of $q(v_{j,k} \pm 2i\gamma)$

To expand $q(v_{j,k} \pm 2i\gamma)$ we write

$$q(v_{j,k} \pm 2i\gamma) = f(v_{j,k} \pm 2i\gamma) \prod_{\substack{l=1\\ l\neq j}}^{n} g_l(v_{j,k} \pm 2i\gamma) \prod_{l=1}^{n_o} h_l(v_{j,k} \pm 2i\gamma) \tag{2.7}$$

with

$$\begin{aligned}
f(v_{j,k} \pm 2i\gamma) &= \prod_{m=1}^{N} \sinh \tfrac{1}{2}(v_{j,k} - v_{j,m} \pm 2i\gamma)\\
g_l(v_{j,k} \pm 2i\gamma) &= \prod_{m=1}^{N} \sinh \tfrac{1}{2}(v_{j,k} - v_{l,m} \pm 2i\gamma) \qquad \text{with} \quad j \neq l\\
h_l(v_{j,k} \pm 2i\gamma) &= \sinh \tfrac{1}{2}\,(v_{j,k} - v^0_l \pm 2i\gamma)
\end{aligned} \tag{2.8}$$

and expand f, g_l and h_l separately. We find

$$
\begin{aligned}
&f(v_{j,k}\pm 2i\gamma)\\
&\quad=\prod_{m=1}^{N}\sinh\tfrac{1}{2}\left(2ik\gamma_0+\epsilon v_{j,k}^{(1)}+\epsilon^2 v_{j,k}^{(2)}-2im\gamma_0-\epsilon v_{j,m}^{(1)}-\epsilon^2 v_{j,m}^{(2)}\pm 2i(\gamma_0+\epsilon)\right)\\
&\quad=\prod_{m=1}^{N}\left(\sinh[i(k-m\pm 1)\gamma_0]\right.\\
&\qquad\left.+[\epsilon(v_{j,k}^{(1)}-v_{j,m}^{(1)}\pm 2i)+\epsilon^2(v_{j,k}^{(2)}-v_{j,m}^{(2)})]\,\tfrac{1}{2}\cosh(i(k-m\pm 1)\gamma_0)\right)\\
&\quad=\tfrac{1}{2}(-1)^{r(N-k\pm 1)}\left(\prod_{m=1}^{N-1}\sinh mi\gamma_0\right)\left(\epsilon(v_{j,k}^{(1)}-v_{j,k\pm 1}^{(1)}\pm 2i)+\epsilon^2(v_{j,k}^{(2)}-v_{j,k\pm 1}^{(2)})\right.\\
&\qquad\left.+(v_{j,k}^{(1)}-v_{j,k\pm 1}^{(1)}\pm 2i)\,\epsilon^2\sum_{l=1}^{N-1}\tfrac{1}{2}(v_{j,k}^{(1)}-v_{j,k\pm(1+l)}^{(1)}\pm 2i)\coth il\gamma_0\right),
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&g_l(v_{j,k}\pm 2i\gamma)\\
&\quad=\prod_{m=1}^{N}\sinh\tfrac{1}{2}(v_{j,k}-v_{l,m}\pm 2i\gamma)\\
&\quad=\prod_{m=1}^{N}\left(\sinh\tfrac{1}{2}(\alpha_j-\alpha_l+2i(k-m\pm 1)\gamma_0+\epsilon[v_{j,k}^{(1)}-v_{l,m}^{(1)}\pm 2i]\right)\\
&\quad=(-1)^{r(N-k\pm 1)}\prod_{m=1}^{N}\sinh\tfrac{1}{2}(\alpha_j-\alpha_l+2im\gamma_0)\\
&\qquad\times\left(1+\epsilon\sum_{m=1}^{N}\tfrac{1}{2}[v_{j,k}^{(1)}-v_{l,m}^{(1)}\pm 2i]\coth\tfrac{1}{2}(\alpha_j-\alpha_l+2i(k-m\pm 1)\gamma_0)\right)
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
h_l(v_{j,k}\pm 2i\gamma)&=\sinh\tfrac{1}{2}\left(\alpha_j-v_l^0+2i(k\pm 1)\gamma_0+\epsilon(v_{j,k}^{(1)}-v_l^{(1)}\pm 2i)\right)\\
&=\sinh\tfrac{1}{2}\left(\alpha_j-v_l^0+2i(k\pm 1)\gamma_0\right)\\
&\qquad\times\left(1+\epsilon(v_{j,k}^{(1)}-v_l^{(1)}\pm 2i)\,\tfrac{1}{2}\coth\tfrac{1}{2}(\alpha_j-v_l^0+2i(k\pm 1)\gamma_0)\right).
\end{aligned}
\tag{2.11}
$$

where we have defined

$$v_{j,N+1}^{(i)}=v_{j,1}^{(i)},\qquad v_{j,N}^{(i)}=v_{j,0}^{(i)}\qquad\text{with}\quad i=1,2. \tag{2.12}$$

Expansion of Eq. (2.2) to Order ϵ

We now are able to expand (2.2). Because $f(v_{j,k} \pm 2i\gamma)$ vanishes as $\gamma \to \gamma_0$ the leading term is of order ϵ. It is convenient to define

$$x_{j,k} = v_{j,k}^{(1)} - v_{j,k+1}^{(1)} + 2i \qquad (-x_{j,k-1} = v_{j,k}^{(1)} - v_{j,k-1}^{(1)} - 2i) \tag{2.13}$$

with

$$\sum_{k=1}^{N} x_{j,k} = 2iN \tag{2.14}$$

and also let

$$\phi_k(\alpha_j) = \sinh^L \tfrac{1}{2}(\alpha_j + (2k-1)\, i\gamma_0) \tag{2.15}$$

where we note the periodicity $\phi_{k+N}(\alpha_k) = \phi_k(\alpha_k)$ Then we find that for each value of j the equation (2.2) reduces to leading order in ϵ to a set of N homogeneous equations for the variables $x_{j,k}$ with $1 \leqslant k \leqslant N$

$$\phi_k(\alpha_j)\, P_{k+1}(\alpha_j)\, x_{j,k} - \phi_{k+1}(\alpha_j)\, P_{k-1}(\alpha_j)\, x_{j,k-1} = 0 \tag{2.16}$$

where we recall the definition of $P_k(\alpha_j)$ of (1.12). The determinant of the coefficients of this set of linear equations vanishes for all values of the still undetermined α_j. Thus equations (2.16) are consistent and with the normalization condition (2.14) we find $x_{j,k}$ in terms of α_j as

$$x_{j,k} = \frac{2iN}{K(\alpha_j)} \frac{\phi_{k+1}(\alpha_j)}{P_k(\alpha_j)\, P_{k+1}(\alpha_j)} \tag{2.17}$$

where

$$K(\alpha_j) = \sum_{k=1}^{N} \frac{\phi_{k+1}(\alpha_j)}{P_k(\alpha_j)\, P_{k+1}(\alpha_j)}. \tag{2.18}$$

We of course have not yet found the desired equations to determine α_j. To do this we need to expand (2.2) to one more order in ϵ.

Expansion of Eq. (2.2) to Order ϵ^2

We now define

$$y_{j,k} = v_{j,k}^{(2)} - v_{j,k+1}^{(2)} \tag{2.19}$$

and find

$$\phi_k(\alpha_j)\, P_{k+1}(\alpha_j)\, y_{j,k} - \phi_{k+1}(\alpha_j) P_{k-1}(\alpha_j)\, y_{j,k-1} = -R_k \tag{2.20}$$

where

$$R_k = R_k^{(1)} + R_k^{(2)} + R_k^{(3)} + R_k^{(4)} \tag{2.21}$$

with

$$\begin{aligned} R_k^{(1)} =& \frac{L}{4}(v_{j,k}^{(1)} - i) \coth\frac{1}{2}[\alpha_j + (2k-1)\, i\gamma_0]\, \phi_k(\alpha_j)\, P_{k+1}(\alpha_j)\, x_{j,k} \\ &- \frac{L}{4}(v_{j,k}^{(1)} + i) \coth\frac{1}{2}[\alpha_j + (2k+1)\, i\gamma_0]\, \phi_{k+1}(\alpha_j) P_{k-1}(\alpha_j)\, x_{j,k-1}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} R_k^{(2)} =& \frac{1}{4}\sum_{l=1}^{N-1} (v_{j,k}^{(1)} - v_{j,k+1+l}^{(1)} + 2i) \coth il\gamma_0 \phi_k(\alpha_j)\, P_{k+1}(\alpha_j)\, x_{j,k} \\ &- \frac{1}{4}\sum_{l=1}^{N-1} (v_{j,k}^{(1)} - v_{j,k-1-l}^{(1)} - 2i) \coth il\gamma_0 \phi_{k+1}(\alpha_j)\, P_{k-1}(\alpha_j)\, x_{j,k-1}, \end{aligned} \tag{2.23}$$

$$\begin{aligned} R_k^{(3)} =& \frac{1}{4}\sum_{\substack{l=1\\ l\neq j}}^{n} \sum_{m=1}^{N} (v_{j,k}^{(1)} - v_{l,m}^{(1)} + 2i) \\ &\times \coth\frac{1}{2}(\alpha_j - \alpha_l + 2(k-m+1)\, i\gamma_0)\, \phi_k(\alpha_j)\, P_{k+1}(\alpha_j)\, x_{j,k} \\ &- \frac{1}{4}\sum_{\substack{l=1\\ l\neq j}}^{n} \sum_{m=1}^{N} (v_{j,k}^{(1)} - v_{l,m}^{(1)} - 2i) \\ &\times \coth\frac{1}{2}(\alpha_j - \alpha_l + 2(k-m-1)\, i\gamma_0)\, \phi_{k+1}(\alpha_j)\, P_{k-1}(\alpha_j)\, x_{j,k-1} \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} R_k^{(4)} =& \tfrac{1}{4}\sum_{l=1}^{n_o} (v_{j,k}^{(1)} - v_l^{(1)} + 2i) \coth\tfrac{1}{2}(\alpha_j - v_l^0 + 2(k+1)\, i\gamma_0)\, \phi_k(\alpha_j)\, P_{k+1}(\alpha_j)\, x_{j,k} \\ &- \tfrac{1}{4}\sum_{l=1}^{n_o} (v_{j,k}^{(1)} - v_l^{(1)} - 2i) \coth\tfrac{1}{2}(\alpha_j - v_l^0 + 2(k-1)\, i\gamma_0)\, \phi_{k+1}(\alpha_j)\, P_{k-1}(\alpha_j)\, x_{j,k-1}. \end{aligned} \tag{2.25}$$

The left hand side of (2.20) is identical with the left hand side of the order ϵ equation (with $x_{j,k} \to y_{j,k}$) and thus the $N \times N$ determinant of the coefficients of $y_{j,k}$ vanishes. This implies that the R_k must satisfy the following constraint

$$\sum_{k=1}^{N} R_k(\alpha_j) \frac{P_k(\alpha_j)}{\phi_k(\alpha_j)\, \phi_{k+1}(\alpha_j)} = 0. \tag{2.26}$$

The Consistency Equation for α_j

It remains to substitute the expression for R_k (2.21) and $x_{j,k}$ (2.17) into (2.26). We consider the four terms $R_k^{(i)}$ separately.

For $R_k^{(1)}$ we find

$$\sum_{k=1}^{N} \frac{R^{(1)}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)\, \phi_{k+1}(\alpha_j)}$$

$$= \frac{L}{4} \sum_{k=1}^{N} \Bigg((v_{j,k}^{(1)} - i) \coth \frac{1}{2} [\alpha_j + (2k-1)\, i\gamma_0] \frac{P_k(\alpha_j)\, P_{k+1}(\alpha_j) x_{j,k}}{\phi_{k+1}}$$

$$- (v_{j,k}^{(1)} + i) \coth \frac{1}{2} [\alpha_j + (2k+1)\, i\gamma_0] \frac{P_{k-1}(\alpha_j)\, P_k(\alpha_j) x_{j,k-1}}{\phi_k} \Bigg) \tag{2.27}$$

$$= \frac{NL}{4} \frac{2i}{K(\alpha_j)} \sum_{k=1}^{N} \Bigg((v_{j,k}^{(1)} - i) \coth \frac{1}{2} [\alpha_j + (2k-1)\, i\gamma_0]$$

$$- (v_{j,k}^{(1)} + i) \coth \frac{1}{2} [\alpha_j + (2k+1)\, i\gamma_0] \Bigg) \tag{2.28}$$

$$= -\frac{iNL}{2K(\alpha_j)} \sum_{k=1}^{N} \coth \frac{1}{2} [\alpha_j + (2k+1)\, i\gamma_0]\, x_{j,k} \tag{2.29}$$

$$= -\frac{iNL}{2K(\alpha_j)} \sum_{k=1}^{N} \coth \frac{1}{2} [\alpha_j + (2k+1)\, i\gamma_0] \frac{2iN\phi_{k+1}(\alpha_j)}{K(\alpha_j) P_k(\alpha_j)\, P_{k+1}(\alpha_j)} \tag{2.30}$$

where to obtain (2.29) we let $k \to k+1$ in the first term of (2.28) and then use the definition of $x_{j,k}$ and to obtain (2.30) we use (2.17). We remark that even though $v_{j,k}^{(1)}$ appears in $R^{(1)}(\alpha_j)$ of (2.22) only the differences $x_{j,k}$ appear in (2.30).

For $R_k^{(2)}$ we find

$$\sum_{k=1}^{N} \frac{R^{(2)}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)\, \phi_{k+1}(\alpha_j)}$$

$$= \frac{1}{4} \sum_{k=1}^{N} \sum_{l=1}^{N-1} \left((v_{j,k}^{(1)} - v_{j,k+1+l} - 2i) \coth il\gamma_0 \frac{P_k(\alpha_j)\, P_{k+1}(\alpha_j)}{\phi_{k+1}(\alpha_j)} x_{j,k} \right.$$

$$\left. - (v_{j,k}^{(1)} - v_{j,k-l-1}^{(1)} - 2i) \coth il\gamma_0 \frac{P_{k-1}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)} x_{j,k-1} \right) \tag{2.31}$$

$$= \frac{iN}{2K(\alpha_j)} \sum_{k=1}^{N} \sum_{l=1}^{N-1} (v_{j,k-1-l}^{(1)} - v_{j,k+1+l}^{(1)} + 4i) \coth il\gamma_0 \tag{2.32}$$

$$= 0 \tag{2.33}$$

where to obtain (2.32) we have used (2.17) and to obtain (2.33) we have let $k \to k + 2(l+1)$ in $v_{j,k-l-1}^{(1)}$ and have used the antisymmetry of $\coth il\gamma_0$ under $l \to N - l$.

For $R_k^{(3)}$ we find

$$\sum_{k=1}^{N} \frac{R^{(3)}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)\, \phi_{k+1}(\alpha_j)}$$

$$= \frac{1}{4} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l \neq j}}^{n} \sum_{m=1}^{N} \left((v_{j,k}^{(1)} - v_{l,m}^{(1)} + 2i) \coth \frac{1}{2} (\alpha_j - \alpha_l + 2i(k-m+1)\, \gamma_0) \right.$$

$$\times \frac{P_k(\alpha_j)\, P_{k+1}(\alpha_j)}{\phi_{k+1}(\alpha_j)} x_{j,k}$$

$$\left. - (v_{j,k}^{(1)} - v_{l,m}^{(1)} - 2i) \coth \frac{1}{2} (\alpha_j - \alpha_l + 2i(k-m-1)\, \gamma_0) \frac{P_{k-1}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)} x_{j,k-1} \right) \tag{2.34}$$

$$= \frac{iN}{2K(\alpha_j)} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l \neq j}}^{n} \sum_{m=1}^{N} \left((v_{j,k}^{(1)} - v_{l,m}^{(1)} + 2i) \coth \frac{1}{2} (\alpha_j - \alpha_l + 2i(k-m+1)\, \gamma_0) \right.$$

$$\left. - (v_{j,k}^{(1)} - v_{l,m}^{(1)} - 2i) \coth \frac{1}{2} (\alpha_j - \alpha_l + 2i(k-m-1)\, \gamma_0) \right) \tag{2.35}$$

$$= -\frac{2N^2}{K(\alpha_j)} \sum_{\substack{l=1 \\ l \neq j}}^{n} \sum_{m=1}^{N} \coth \frac{1}{2} (\alpha_j - \alpha_l + 2im\gamma_0). \tag{2.36}$$

We note that even though $v_{j,k}^{(1)}$ appears in $R^{(3)}(\alpha_j)$ it has canceled in the final expression (2.36).

Finally for $R_k^{(4)}$ we have

$$\sum_{k=1}^{N} \frac{R^{(4)}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)\, \phi_{k+1}(\alpha_j)}$$

$$= \frac{1}{4} \sum_{k=1}^{N} \sum_{l=1}^{n_o} \bigg((v_{j,k}^{(1)} - v_l^{(1)} + 2i) \coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k+1)\, i\gamma_0) \frac{P_k(\alpha_j)\, P_{k+1}(\alpha_j)}{\phi_{k+1}(\alpha_j)} x_{j,k}$$

$$- (v_{j,k}^{(1)} - v_l^{(1)} - 2i) \coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k-1)\, i\gamma_0) \frac{P_{k-1}(\alpha_j)\, P_k(\alpha_j)}{\phi_k(\alpha_j)} x_{j,k} \bigg) \tag{2.37}$$

$$= \frac{iN}{2K(\alpha_j)} \sum_{k=1}^{N} \sum_{l=1}^{n_o} \bigg((v_{j,k}^{(1)} - v_l^{(1)} + 2i) \coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k+1)\, i\gamma_0)$$

$$- (v_{j,k}^{(1)} - v_l^{(1)} - 2i) \coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k-1)\, i\gamma_0) \bigg) \tag{2.38}$$

$$= \frac{iN}{2K(\alpha_j)} \sum_{k=1}^{N} \sum_{l=1}^{n_o} \bigg(\coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k+1)\, i\gamma_0)$$

$$+ \coth \frac{1}{2} (\alpha_j - v_l^0 + 2ki\gamma_0) \bigg) x_{j,k} \tag{2.39}$$

$$= \frac{iN}{2K(\alpha_j)} \sum_{k=1}^{N} \sum_{l=1}^{n_o} \bigg(\coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k+1)\, i\gamma_0)$$

$$+ \coth \frac{1}{2} (\alpha_j - v_l^0 + 2ki\gamma_0) \bigg) \frac{2iN}{K(\alpha_j)} \frac{\phi_{k+1}(\alpha_j)}{P_k(\alpha_j)\, P_{k+1}(\alpha_j)} \tag{2.40}$$

where we note that $v_l^{(1)}$ (for which we have not derived an expression) has canceled out in going from (2.38) to (2.39).

Thus we may use (2.30), (2.33), (2.36) and (2.40) in (2.26) to obtain

$$\frac{1}{K(\alpha_j)} \sum_{k=1}^{N} \frac{\phi_{k+1}(\alpha_j)}{P_k(\alpha_j)\, P_{k+1}(\alpha_j)} \bigg\{ L \coth \frac{1}{2} (\alpha_j + (2k+1)\, i\gamma_0)$$

$$- \sum_{l=1}^{n_o} \bigg(\coth \frac{1}{2} (\alpha_j - v_l^0 + 2(k+1)\, i\gamma_0) + \coth \frac{1}{2} (\alpha_j - v_l^0 + 2ki\gamma_0) \bigg) \bigg\}$$

$$- 2 \sum_{\substack{l=1 \\ l \neq j}}^{n} \sum_{m=1}^{N} \coth \frac{1}{2} (\alpha_j - \alpha_l + 2im\gamma_0) = 0. \tag{2.41}$$

The result (1.11) of the introduction follows immediately.

3. DISCUSSION

The equation (1.9) for the ordinary roots v_l^0 and the equation (1.11) for α_k replace the Bethe's equation (1.2) at roots of unity where undefined factors of $0/0$ occurred. Thus for the case of $\gamma = r\pi/N$ we have succeeded for the first time in completely specifying the parameters v_j which occur in the Bethe's Ansatz wave function as given by Yang and Yang[5] and in the auxiliary matrix $Q(v)$ in Baxter's[29, 34] functional equation for the transfer matrix of the six vertex model. We have also numerically verified for the cases $N = 2-5$, $r = 1$ that the equations (1.9) and (1.11) reproduce the numerical results previously obtained by other means in ref. 31.

There are several other features of our computation which deserve to be discussed in detail.

First of all even though we have expanded (2.2) to order ϵ^2 and have found an explicit solution (2.17) for the variable $x_{j,k}$ (2.13) which depends on the differences $v_{j,k}^{(1)} - v_{j,k+1}^{(1)}$ we have not obtained expressions for the first order corrections $v_{j,k}^{(1)}$ themselves which will in general contain a constant independent of k which is not present in $x_{j,k}$. This constant plays a role in the first order correction identical to the role which α_j plays in the zeroth order solution and is determined from the consistency equation needed for the determination of the ϵ^3 corrections. This pattern of the consistency equation for order ϵ^{n+2} being needed to completely specify the roots to order ϵ^n is a general feature for all orders in ϵ.

Secondly we point out that the parameters α introduced in (1.5) as part of the specification of the complete exact N string do not have to be real and that if α is complex then we see from taking the complex conjugate of (2.41) that α^* will also be a a solution (because all ordinary roots appear in complex conjugate pairs). These complex values of α are a new feature of the solution of Bethe's equation which were first seen in our[31] previous numerical treatment of the problem where many examples were exhibited.

Thirdly we acknowledge that to completely solve the eigenvalue degeneracy problem we need to be able to classify and count all solutions of (1.9) and (1.11). We studied this degeneracy in refs. 30 and 31 in terms of the finite dimensional representations of the sl_2 loop algebra and we remarked that from the theory of affine Lie algebras[33] it is known that these representations are specified by what are called "evaluation" parameters and that these parameters are roots of the Drinfeld polynomial. However these evaluation parameters are not the same as the roots α_j studied in this paper and we note in particular that for a multiplet containing 2^m degenerate eigenvalues there are only m evaluation parameters even though there are 2^m solutions for α_j all of which have the same n_o ordinary roots. Thus it would seem that there is a sense in which there is more information in the α_j than what is needed for the solution of the degeneracy problem for the eigenvalues of the Hamiltonian (and the transfer matrix).

To explore further the nature of the information contained in the evaluation parameters we have explicitly computed them for a chain of $L = 12$ and $N = 3$ and have found that the evaluation parameters of each multiplet are different. This means that there are no isomorphisms between multiplets and thus we reach the striking conclusion for this case (and we believe for all cases with $N \geqslant 3$) that the sl_2 loop algebra is not only a symmetry algebra which classifies states into degenerate multiplets but that it is sufficiently powerful to determine all the highest weight vectors as well. Therefore the complete decomposition of the representation of the loop sl_2 algebra into finite dimensional irreducible representations must produce as highest weight vectors the same vectors which are determined by the Bethe's equation (1.2) when used in the Bethe form of the wave function.[5] This relation between finite dimensional irreducible representations of the loop algebra of sl_2 and Bethe's ansatz has not been previously noticed.

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